Tiling on multipartite graphs

Ryan Martin

Mathematics Department
Iowa State University
rymartin@iastate.edu

Texas State Discrete Math Seminar

This talk includes joint work with:

- Csaba Magyar
- Endre Szemerédi, Rutgers University and the Rényi Institute
- Yi Zhao, Georgia State University

The Hajnal-Szemerédi theorem

Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If G is a simple graph on n vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n$$

then G contains a subgraph which consists of $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r.

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- $r = 3$ proven by Corrádi & Hajnal 1963
- New proof by Kierstead & Kostochka 2008 (discharging)

The Alon-Yuster theorem

Theorem (Alon-Yuster, 1992)

For any $\alpha > 0$ and graph H, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph G on $n \geq n_0$ vertices with

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Komlós, Sárközy and Szemerédi, 2001, showed that αn can be replaced by $C = C(H)$, but not eliminated entirely.

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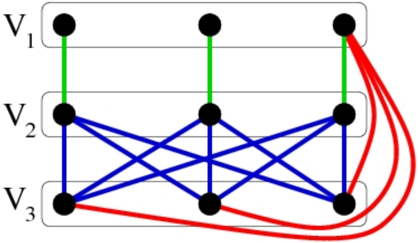
Definition

The natural bipartite subgraphs of G are the ones induced by the pairs of classes of the r-partition.

Natural bipartite subgraphs

Example Consider the graph G:



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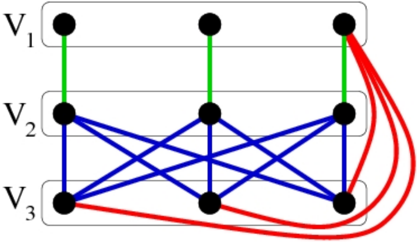

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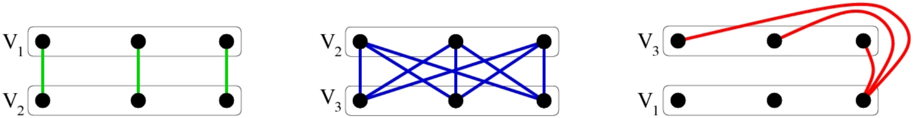

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Conjecture [Fischer]

If $G \in \mathcal{G}_r(N)$ and

$$\bar{\delta}(G) \geq \left(1 - \frac{1}{r}\right) N$$

then G has a K_r-factor.

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Let $G \in \mathcal{G}_r(N)$ and $\bar{\delta}(G) \geq \left(1 - \frac{1}{r}\right) N$. Then,

$$\delta(G) \quad \geq \quad (r-1)\left(1 - \frac{1}{r}\right) N$$

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$$\begin{aligned} \delta(G) &\geq (r-1)\left(1 - \tfrac{1}{r}\right) N \\ &= \left(\tfrac{r-1}{r}\right)^2 (rN) \end{aligned}$$

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So the total degree is not large enough to invoke Hajnal-Szemerédi.

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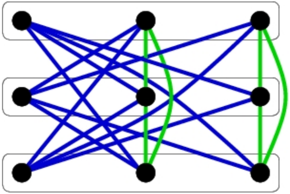
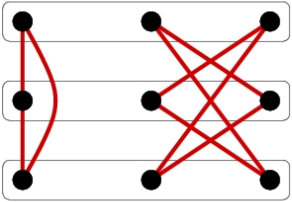
Example Let $r = 3$ and $N = 3$:



General Example

Redraw the example with nonedges:



$\Gamma_3(3)$

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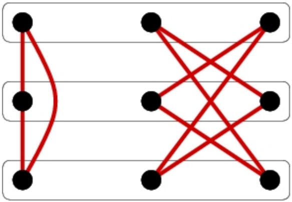

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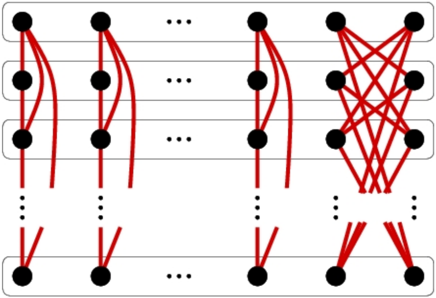

$\Gamma_r(r)$

This complement can be attributed to Paul Catlin, 1976, and was called a "type 2 graph."

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- Replace each vertex with N/r vertices.
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Then, $\Gamma_r(N) \in \mathcal{G}_r(N)$.

If $r \mid N$, then $\Gamma_r(N/r)$ has no K_r-factor iff r is odd and N/r is odd.

Tripartite theorem

Theorem (Magyar-M, 2002)

There exists an N_0 *such that if* $N \geq N_0$, $G \in \mathcal{G}_3(N)$ *and*

$$\bar{\delta}(G) \geq \frac{2}{3}N,$$

then G *has a* K_3*-factor unless*

$G \approx \Gamma_3(N)$ *and* $N/3$ *is an odd integer.*

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(N need not be divisible by 3.)

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Theorem (M-Szemerédi, 2008)

There exists an N_0 such that if $N \geq N_0$, $G \in \mathcal{G}_4(N)$ and

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There is no exceptional graph.

Bipartite graph factors

Theorem (Zhao, 2009)

Let h be a positive integer. There exists an $N_0 = N_0(h)$ such that if $N \geq N_0$, $h \mid N$, and $G \in \mathcal{G}_2(N)$ with

$$\bar{\delta}(G) \geq \begin{cases} \frac{N}{2} + h - 1, & N/h \text{ is odd;} \\ \frac{N}{2} + \frac{3h}{2} - 2, & N/h \text{ is even,} \end{cases}$$

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then G has a $K_{h,h}$-factor.

Moreover, there are examples that prove that this $\bar{\delta}$ condition cannot be improved.

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If $\chi(H) = 2$ and $|V(H)| = h$, then $K_{h,h}$-factor $\Rightarrow$ H-factor.

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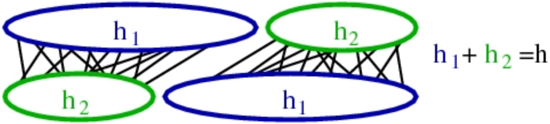
Example.



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implies G has a $K_{h,h,h}$*-factor. Then*

$$\begin{array}{rcccl} & & f(h) & = & h-1, \quad \text{if } N/h \equiv 0 \bmod 6; \\ h-2 & \leq & f(h) & \leq & h-1, \quad \text{if } N/h \not\equiv 0 \bmod 3; \\ h-1 & \leq & f(h) & \leq & 2h-1, \quad \text{if } N/h \equiv 3 \bmod 6. \end{array}$$

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Note

Both $\chi(H) = 3$ and $|V(H)| = h$ together imply a H-factor also.

Other graph factors

The case analysis required to prove that $\bar{\delta}(G) \geq (3/4 + \epsilon)N$ is sufficient for a $K_{h,h,h,h}$-factor would be long and difficult, using current methods. However, we believe it could be done.

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To prove the existence of an $f(h)$ would be even more difficult.

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For every $\Delta > 0$, there exists an $\epsilon > 0$ such that if $G \in \mathcal{G}_3(N)$,

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- $\exists$ a partial K_3-factor $\mathcal{T}'$ with $|\mathcal{T}'| > |\mathcal{T}_0|$ and $|\mathcal{T}' \setminus \mathcal{T}_0| \leq 15$ or
- $\exists$ 3 sets which are each of size $N/3$ but have pairwise density $\leq \Delta$.

Best general bound

Theorem (Csaba-Mydlarz, 2009+)

Let $r \geq 5$ *and* $\epsilon > 0$. *There exists an* $N_0 = N_0(r, \epsilon)$ *such that if* $N \geq N_0$, $G \in \mathcal{G}_r(N)$ *and if*

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This is the best bound for $r \geq 5$.

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$\chi_{\rm cr}(H)$ was defined by Komlós, 2000.

Use of critical chromatic number

Theorem (Komlós, 2000)

For every H and every n, divisible by $|V(H)|$, there exists a G of order n with

$$\delta(G) = \left\lceil \left(1 - \frac{1}{\chi_{\rm cr}(H)}\right) n \right\rceil - 1$$

and no H-tiling.

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$$\delta(G) \geq \left(1 - \frac{1}{\chi_{\mathrm{cr}}(H)}\right) n$$

then G has an H-tiling that covers all but ϵn vertices in G.

Use of critical chromatic number

Theorem (Komlós, 2000)

For every H and every n, divisible by $|V(H)|$, there exists a G of order n with

$$\delta(G) = \left\lceil \left(1 - \frac{1}{\chi_{\rm cr}(H)}\right) n \right\rceil - 1$$

and no H-tiling.

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Question

Does χ_{cr} provide a better minimum-degree parameter for finding an H-tiling of an r-partite graph where $r = \chi(H)$?

Possible solution techniques

- Ideas from the Kierstead-Kostochka proof

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  - there is a structure that might be modified to apply their main lemma.

Open problems

- Is it true that $\exists C$ such that $G \in \mathcal{G}_5(N)$ and $\bar{\delta}(G) \geq (4/5)N$ implies that there exists a partial K_5-tiling of size $(1-\epsilon)N$?

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- Is it true that, $\forall \epsilon > 0$, $\exists N_0 = N_0(\epsilon)$ such that $N \geq N_0$, $G \in \mathcal{G}_5(N)$ and $\bar{\delta}(G) \geq (4/5 + \epsilon)N$ implies a K_5-tiling?

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Almost-covering question ($r = 5$)

Does there exist an absolute constant C such that:
For all $\epsilon > 0$, if $G \in \mathcal{G}_5(N)$,

$$\bar{\delta}(G) \geq \left(\frac{4}{5} + \epsilon\right) N$$

and $\mathcal{T}_0$ is a partial K_5-factor of G with $\mathcal{T}_0 < N - C$, then $\exists$ a partial K_5-factor $\mathcal{T}'$ with $|\mathcal{T}'| > |\mathcal{T}_0|$?

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I.e., $(1/2 + \epsilon)N$ is sufficient. What about $(1 - 1/\chi_{\mathrm{cr}}(H) + \epsilon)N$?

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